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Characterizations of Selfadjoint Open Central Tripotents in *-*TRO*

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Abstract

This paper gives a number of characterizations of open central tripotents, and lists some of their basic properties. Another useful way of looking at open tripotents is also provided.

Key words: Tripotent, weak *, *-isomorphism, *-automorphism and *-ideal.

1. Introduction

Blecher and Werner[3] investigated some order structures on C*-bimodules. Hamana[5] also developed the ternary rings of operators. Akemann[1] introduced the 'tripotent' variant of an open projection. Hence this paper will discuss about the characterization of the possible orderings on a C*-bimodule which correspond to embeddings of this C*-bimodule as selfadjoint ternary rings of operators. This paper's characterization is in terms of ***selfadjoint tripotents***; namely elements u such that $u = u^* = u^3$. This paper also studieds the basic properties associated with the Akemann's tripotent notion (e.g. [1]).

Now, we are in a position to precise definitions and notation. Most of the definitions and notations will be omitted because we no need to delay. We refer [3] for the omitted definitions and notation. We write X_+ for the cone of 'positive elements', i.e. those with $x \geq 0$, in an ordered vector space. A ***ternary rings of operators*** (or ***TRO*** for short) is a closed subspace X of a C*-algebra B such that $XX^*X \subset X$. We will assume Hemana's ***ternary system*** and *TRO* are equivalent (e.g. [3], Proposition 1.1 [3]). A closed selfadjoint *TRO* Z in a C*-algebra B is a ****-TRO***. For a subspace A of a C*-algebra B, $A^\perp$ is the ***annihilator*** of A. X' for the dual Banach space of a Banach space X. We have used '*' above for the 'adjoint' or 'involution'.

Let $S(E)$ be the set of selfadjoint tripotents in the centre $Z(E)$ of E. If u, v are selfadjoint tripotents in $S(E)$, then the greatest lower bound of $\{u, v\}$ (this g.l.b exists by [3]) in $S(E)$ is

$$u \wedge v = \frac{1}{2}(uvu + vuv). \tag{1.1}$$

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If E is a *-*TRO*, and if u is a selfadjoint tripotent in $Z(E)$, then the cone $\mathfrak{c}_u$ (as defined in [3], definition 4.6) is equal to $\{eue^* : e \in E\}$, which is also equal to $\{x \in E : ux \in (E^2)_+, x = uxu\}$. Let E be an involutive ternary system. A given cone E_+ in E (resp. $M_n(E)_+ \subset M_n(E)$ for all $n \in N$) is a natural dual cone (resp. are natural dual matrix cones) if and only if for some selfadjoint tripotent $u \in Z(E)$, $E_+ = \{eue^* : e \in E\}$ (resp. $M_n(E)_+ = \{[\sum_k e_{ik}ue^*_{jk} : [e_{ij}] \in M_n(E)\}, n \in \mathrm{N}$).

The two propositions show the important characterizations of ordering structures on involutive ternary system whose proof can be found in [3].

Proposition(1.1) Let E be a dual involutive ternary system. Then there is a bijective correspondence between natural dual orderings on E and selfadjoint tipotents of $Z(E)$. This correspondence respects the ordering of cones by inclusion on one hand, and the ordering of selfadjoint tripotents given by $u_1 \leq u_2$ if and only if $u_1u_2u_1 = u_1$.

Proposition(1.2) Let E be a dual involutive ternary system. If $v \in Z(E)$ is a selfadjoint tripotent, then the following conditions are equivalent:

(a) v is an extreme point of $Z(E)_{sa}$.
(b) v is a selfadjoint unitary of $Z(E)$.
(c) v is maximal with respect to the order defined in Proposition (1.1).
(d) $Z(E) \subseteq J(v)$.
(e) $J(v)$ contains every selfadjoint unitary of $Z(E)$.

2. Selfadjoint Open Central Tripotents in *-*TRO*

In this section, we will give several characterizations of open central tripotents, and list some of their basic properties. First we give the definition of an open tripotent.

Definition(2.1) Let Z be an involutive ternary system, and let u be a selfadjoint tripotent in $Z(Z'')$. We define $\mathfrak{d}_u$ be the cone $\mathfrak{c}_u \cap Z$ in Z. We also write $J_u(Z)$ for the span of $\mathfrak{d}_u$ in Z, and $\mathfrak{C}_u$ for the weak* closure of $\mathfrak{d}_u$ in Z''. We say that a tripotent u is open if, when we consider $J(u)$ as a W*-algebra in the canonical way, then u is the weak* limit in Z'' of an increasing net in $J(u)_+ \cap Z$.

Note that an open projection in the second dual of a C*-algebra is open in the usual sense if and only if it is an open tripotent in the sense above.

Lemma(2.2) Let Z be an involutive ternary system, and let $E = Z''$, also an involutive ternary system in the canonical way. Let u be a selfadjoint tripotent in Z(E). Then

(1) $\mathfrak{c}'_u \subset \mathfrak{c}_u$.

(2) $\mathfrak{d}_u$ is a natural cone on Z. That is, there is a surjective ternary order *-isomorphism from Z with cone $\mathfrak{d}_u$ to W, for a *-*TRO* W. Also ψ'' is a ternary order *-isomorphism from Z'' with cone $\mathfrak{c}'_u$, onto W'' with its canonical second dual cone.

(3) $\mathfrak{c}'_u = \mathfrak{c}_v$ for a selfadjoint tripotent $v \in Z(E)$ with $v \leq u$.

(4) $J_u(Z)$ is a C*-subalgebra of $J(u)$, when we regard $J(u)$ as a C*-algebra in the canonical way. Also, $\mathfrak{d}_u$ is the positive cone of $J_u(Z)$.

(5) If Z is a *-*TRO* in a C*-algebra B, then $u\mathfrak{d}_u \subset \mathfrak{d}_u^2 \subset J_u(Z)^2 \subset Z^2$. Thus $uJ_u(Z) \subset J_u(Z)^2$.

Proof. (1) By the properties in the last paragraph of our introduction, E may be regarded as a *-*WTRO* whose positive cone is $\mathfrak{c}_u$, and this is weak* closed in E. Then $\mathfrak{c}'_u = \overline{\mathfrak{d}_u}^{w*} \subset \mathfrak{c}_u$.

(2) By regarding Z as a *-*TRO* inside E, $\mathfrak{d}_u$ is just the inherited cone. Thus it is a natural cone. Since $J(Z)$ is weak* dense in $J(E)$ and by Kaplansky's density theorem, $\mathfrak{c}_u$ coincides with the canonical second dual cone induced from $\mathfrak{d}_u$.

(3) From the last part of (2), E_+ is the natural dual cone. Thus

$$E_+ = \{eue^* \,;\, e \in E\}.$$

By Proposition, (3) follows.

(4) Since $J(Z_+) = Z_+$, and by (3), we get (4).

(5) Let $x \in \mathfrak{d}_u$. Then $x = yuy$ for some $y \in \mathfrak{d}_u$. Thus

$$ux = u^2yy = yy \in \mathfrak{d}_u^2 \subset J_u(Z)^2.$$

Then (5) follows. +

Theorem 1 Let Z be an involutive ternary system. Then

(1) The natural cones on Z are precisely the cones $\mathfrak{d}_u = \mathfrak{c}_u \cap Z$, for u a selfadjoint open tripotent in $Z(Z'')$. A similar fact holds for the natural matrix cones on Z.

(2) There is a bijective correspondence between natural orderings on Z and selfadjoint open tripotents in $Z(Z'')$. This correspondence respects the ordering of the cones by inclusion on one hand, and the ordering of selfadjoint tripotents given by $u_1 \leq u_2$ if and only if $u_1u_2u_1 = u_1$.

Proof. (1) If Z is a *-*TRO*, then so is $E = Z''$. Applying last paragraph of introduction to the canonical ordering on E yields a selfadjoint tripotent u with $J(u) = J(E)$, and $E_+ = \mathfrak{c}_u$.

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That u, the identity of the W*-algebra $J(E)$, is open since $J(Z)$ is weak* dense in $J(E)$. Thus Z_+ has the cone $\mathfrak{d}_u$. A similar argument applies to $M_n(Z)_+$.

The converse follows from no:(2) of Lemma 2.2.

(2) Suppose that u, v are two selfadjoint open tripotents in $Z(E'')$. Set

$$A = \{eue^* : e \in E''\} \cap E, \text{ and } B = \{eve^* : e \in E''\} \cap E.$$

Suppose first that $A \subset B$. By definition u is a weak* limit of a net in A, and hence in B. Since $\mathfrak{c}_u$ is weak* closed (being the cone of a W*-algebra), and we have that u is in $\mathfrak{c}_v$. It follows as in the proof of Proposition 1.1, that $u \leq v$. Conversely, if $u \leq v$, then the second part of Proposition 1.1 shows that $A \subset B$. +

We now give some equivalent, and often more useful, characterizations of selfadjoint open central tripotents. But first we will need one definition and some remarks.

Definition(2.2) Let B be a C*-algebra and J be a C*-ideal in an involutive ternary system Z, with $\psi : B \to J$ the surjective triple *-isomorphism. A selfadjoint tripotint u in $Z(Z'')$ such that $\psi''(1) = u$ is called a support tripotent for J.

Remark.

(1) If (e_t) is an increasing approximate identity for B, then it is well known that $e_t \to 1_{B''}$ weak*. Thus $(e_t) \to u$ weak*, and hence u is an open tripotent in the sense above.

(2) J with the product pulled back from B via ψ is a C*-subalgebra of $J(u)$, the latter with its canonical product(see [3; Corollary 4.2]). Clearly $J = J^{\perp\perp} \cap Z = J(u) \cap Z$

(3) The induced natural cone $\mathfrak{d}_u$ on Z (see Definition 2.1), equal $\psi(B_+)$, the canonical positive cone for J. To see this note that

$$\psi(B_+) = \psi''(B''_+ \cap B) = \psi''(B''_+) \cap J = \mathfrak{c}_u \cap (J(u) \cap Z) = \mathfrak{d}_u.$$

Theorem2 Let Z be a *-*TRO*, and let $E = Z''$. Let u be a selfadjoint tripotent in $Z(E)$. The following are equivalent:

(i) u is an open tripotent.

(ii) $u \in \mathfrak{d}_u$.

(iii) There is a net (x_i) in Z converging weak* to u, satisfying : $ux_i \geq 0$, $u^2 x_i = x_t$ for all t, and (ux_i) is an increasing net.

(iv) If $J_u(Z)$ is the span of the cone $\mathfrak{d}_u$ in Z, then $\overline{J_u(Z)}^{w^*} = J(u)$.

(v) u is a support tripotent for a C*-ideal in Z.

(vi) u^2 is an open projection (in the usual sense) in $(Z^2)''$, and $u(J(u) \cap Z) \subset Z^2$.

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(vii) $\mathcal{C}'_u = \mathcal{C}_u$.

Proof. By Definition 2.1, (i) is equivalent to (iii). It is clear that (iii) implies (ii), and (vii) implies (ii). We now show that (ii) implies (vii), (i) and (iv). Using Lemma 2.2, and the first part of the proof of Theorem 1, we have $\mathcal{C}'_u = \mathcal{C}_v$ for a selfadjoint open central tripotent v. Thus if (ii) holds then $u \in \mathcal{C}_v$, so that $\mathcal{C}_u \subset \mathcal{C}_v = \mathcal{C}'_u$. But by Lemma 2.2 again, $\mathcal{C}'_u \subset \mathcal{C}_u$. So $\mathcal{C}_u = \mathcal{C}'_u = \mathcal{C}_v$. Hence $u = v$ is open. by ([3; Propositon 3.3 (3) and Lemma4.1]), $\overline{J_u(Z)}^{w*} = J(u)$. Thus we have verified (vii), (i) and (iv).

(iv) $\Rightarrow$ (i) By the proof of Lemma 2.2(5), we have that $J_u(Z)$ is a C*-subalgebra of $J(u)$. Then (i) follows by Kaplansky's density theorem.

(iv) $\Rightarrow$ (vi) Note that $J(u) \cap Z = J_u(Z)$. Hence by Lemma 2.2(5),

$$u(J(u) \cap Z) = uJ_u(Z) \subset J_u(Z)^2 = (J(u) \cap Z)^2 \subset Z^2.$$

Since (iv) implies (i), u^2 is a weak* limit of an increasing net in $u(J(u) \cap Z) \subset Z^2$. Thus u^2 is open.

(v) $\Rightarrow$ (i) If (e_t) is an increasing approximate identity for B, then it is well-known that $e_t \to 1_{B''}$ weak*. Thus $\psi(e_t) \to u$ weak*, and hence u is an open tripotent.

(vi) $\Rightarrow$ (v) Let $J = J(u) \cap Z$, this is a ternary *-ideal in Z, and if (vi) holds then J is a C*-ideal in Z. Indeed J is a C*-subalgebra of the W*-algebra $J(u)$. By Remark 2.5 (1), if (e_t) is an increasing approximate identity for J, then $e_t \to v$ weak* in $J(u)$, where v is a support tripotent for J. Clearly $vux = x$ for all x 2 J, and hence by weak* density we have $vuv = v$. Also by Remark no:(2), $J = J(v) \cap Z$. Now v and v^2 are open (since (v) implies (i) and (vi)), and ([3; Lemma 3.4(2)]) gives $u^2 = v^2$. Thus $vuv = u = v$. +

We now give another useful way of looking at open tripotents.

Proposition(2.1) Let Z be a *-*TRO*, and let $A = Z + Z^2$.

(1) Suppose that u is a selfadjoint open tripotent in $Z(Z'')$. Then $p = \dfrac{u^2 + u}{2}$ and $q = \dfrac{u^2 - u}{2}$ are open projections in the center of A''. Moreover, $u = p - q$ and $pq = 0$.

(2) Suppose that $Z \cap Z^2 = \{0\}$ (this may be always be ensured, by replacing Z by a ternary *-isomorphic *-*TRO*). Let $\theta : A \to A$ be the period 2 *-automorphism $\theta(z + a) = a - z$ for a $\in Z^2$, $z \in Z$. Suppose that p is an open projection in the center

of A'', such that $pq = 0$, where $q = \theta''(p)$. Then $u = p - q$ is a selfadjoint open tripotent in $Z(Z'')$, and q is also an open projection in the center of A''.

Proof. (1) By direct calculation, we have p, q are projections in the center of A'', and that $u = p - q$ and $pq = 0$. To show that p, q are open, let (x_t) be the net in no:(iii) of Theorem 2. Then $ux_t \in Z^2$ by no: (5) of Lemma 2.2. Thus $\frac{ux_t + x_t}{2}$ is a net in A converging weak* to p, and also $p\frac{ux_t + x_t}{2} = \frac{ux_t + x_t}{2}$. It follows that p is an open projection. Similarly for q.
(2) It is clear that u is a tripotent in $Z(A'')$. Also θ'' is a **period** 2 *-automorphism of A''. Suppose that (a_t) is an increasing net in A converging weak* to p, with $pa_t = a_t$. Then $b_t = \theta(a_t)$ is a net in A converging weak* to $\theta''(p)$, and $\theta''(p)b_t = b_t$. Also $\theta(a_t - b_t) = b_t - a_t$. Hence $y_t = a_t - b_t$ is a net in Z converging weak* to u . So $u \in Z(Z'')$. Clearly $u^2y_t = (p + \theta''(p))$
$(a_t - b_t) = y_t$. Moreover, uy_s is the weak* limit of

$$(a_t - \theta(a_t))(a_s - \theta(a_s)) = a_ta_s + \theta(a_ta_s),$$

since $\theta(a_t)a_s = \theta(a_t)\theta''(p)pa_s = 0$. Thus $uy_s = as + \theta(as) \geq 0$. Thus no: (ii) of Theorem 2 implies that $u = p - q$ is a selfadjoint open tripotent. Since θ'' is a weak* continuous *-automorphism, q is an open projection in the center of A''. +

We end this paper with some discussions. The intersection of natural cones on a *-*TRO* is again a natural cone, as may be seen by the discussion in [3]. Hence every family of selfadjoint open tripotents has a greatest lower bound amongst the selfadjoint open tripotents. The following fact is a little deeper, and should be important in future developments. It is the 'tripotent version' of Akemann's result that the infimum of two (in this case central) open projections is open.

Corollary Let Z be a *-*TRO*, and let u, v be two selfadjoint open tripotents in $Z(Z'')$. Then the greatest tripotent $u \wedge v$ in $Z(Z'')$ majorized by u and v is open. Also, $\mathfrak{d}_u \cap \mathfrak{d}_v = \mathfrak{d}_{u\wedge v}$.

Proof. The last assertion is clear:

$$\mathfrak{d}_u \cap \mathfrak{d}_v = \mathfrak{c}_u \cap \mathfrak{c}_v \cap Z = \mathfrak{c}_{u\wedge v} \cap Z = \mathfrak{d}_{u\wedge v}.$$

To obtain the other fact, we appeal to no:(2) of Proposition 2.1. We check that $u \wedge v$ is open.
By calculation,

$$\frac{(u\wedge v)^2 + u\wedge v}{2} = \frac{u^2 + u}{2}\frac{v^2 + v}{2}.$$

The right side of the above equation is a product of two commuting open projections (by no:(1) of Proposition 2.1), and hence is open.

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